Tachyonic Instability and Darboux Transformation

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Abstract

Using Darboux transformation one can construct infinite family of potentials which lead to the flat spectrum of scalar field fluctuations with arbitrary multiple precision, and, at the same time, with "essentially blue" spectrum of perturbations of metric. Besides, we describe reconstruction problem: find classical potential $V(\phi)$ starting from the known "one-loop potential" $u(t) = d^2V(\phi(t))/d\phi(t)^2$.

1 Introduction.

In [1] a new mechanism for the generation density perturbation by tachyonic instability is discussed. If we assume that

1. the universe was flat and homogeneous;
2. Hubble constant is small during the process generation of the perturbation

then the equation for the quantum fluctuations $\delta\phi_k(t)e^{-i\mathbf{kr}}$ of scalar field is

$$\delta\ddot{\phi}_k + \left(k^2 + u(t)\right)\delta\phi_k = 0, \tag{1}$$

where $u(t) = d^2V(\phi(t))/d\phi(t)^2$ (we call $u(t)$ as "one-loop potential"), $V(\phi)$ is classical potential, $\phi(t)$ is a homogeneous solution of classical equation of motion. To calculate spectrum of density one need to find solution of (1) with the asymptotic behavior $\delta\phi_k(t \to +\infty) = e^{ikt}/\sqrt{2k}$. Then density fluctuations at moment t_0 are given by

$$\delta_k \sim H\left(\frac{k}{2\pi}\right)^{3/2}\frac{\delta\phi_k(t_0)}{\delta\phi_0(t_0)}, \qquad n_s = 1 + \frac{d\log\mid\delta_k\mid^2}{d\log k}, \tag{2}$$

where $\delta\phi_0$ is solution of the (1) with $k=0$, n_s is the spectral index deciding the "color" of spectrum. It is easy to see that $\delta\phi_0$ is both $k=0$ solution of the (1) and the time derivative of the field $\phi(t)$: $\delta\phi_0(t) = \dot{\phi}(t)$ so it's dimensions is $[\delta\phi_0]$ =GeV^2 whereas for the $\delta\phi_k(t)$ we choose the normalization factor to have $[\delta\phi_k]$ =GeV$^{1/2}$ (in the units with $c = \hbar = 1$).

There are just two usually studied potentials: power-law potential $V(\phi) = -\lambda\phi^n/M^{n-4}$, ($M$ is the some constant, $\lambda > 0$ [1]) and "ekpyrotic" or "cyclic" potential $V(\phi) = -V_0e^{-\phi/M}$

[1] The potential $V(\phi) < 0$ so one needs to add to $V(\phi)$ some other terms to stabilize the motion. But it doesn't matter in this paper which is devoted to tachyonic instability.

[2, 3]. Both of them leads to the equation (1) with $u(t) = -\mu^2/t^2$. For the power-law potentials $\mu^2 = \mu^2(n) = 2n(n-1)/(n-2)^2$ and it easy to see that all quantities $\mu^2(n) > 0$ can be obtained by appropriate choosing of n. For example, if $n > 2$ then $\mu^2(n) \in (2, +\infty)$; if $n \in [1, 4/3]$ then $\mu^2(n) \in [0, 2]$ and for the $n \in (-\infty, 0]$ we have $\mu^2(n) \in [0, 2)$. For the "cyclic" potential $\mu^2 = 2$.

In these both cases the equation (1) can be solved via Hankel function. The aim of this work to call attention to simple but effective method to construct infinite set of one-loop potentials with two properties:

1. All this one-loop potentials are integrable. This allow one to find $\delta\phi_k$ and $\delta\phi_0$ exactly to calculate δ_k.
2. These one-loop potentials lead to almost flat spectrum.

Although, our approach get us a family "one-loop" potentials $u(t)$ one can reconstruct initial classical potential $V(\phi)$ starting out from the $u(t)$. We do it in the Sec. 3. In this next we also discuss the compatibility one of new potential with inflation. One show that this potential can lead to inflation but for very specific initial condition. This fact is at one with analysis in [1].

Our results about the flat spectrum are valid only for the spectrum of the scalar field and, to all appearances, don't valid for the spectrum of density perturbations. In Sec. 4, we show that using Darboux transformations, one can construct infinite set exact soluble cosmologies with flat spectrum of the scalar field perturbations and, at the same time, with "essentially nonflat", blue spectrum of perturbations of metric.

2 Flat spectrum via Darboux transformation.

Let consider the equations

$$\delta\ddot{\phi}_k + (k^2 + u(t))\delta\phi_k = \delta\ddot{\psi}_\kappa + (-\kappa^2 + u(t))\delta\psi_\kappa = 0, \tag{3}$$

where $u(t) = -2/t^2$, k and κ are real numbers. One choose the solution of the first equation (3) as ($t > 0$).

$$\delta\phi_k = N\left(1 + \frac{i}{kt}\right)e^{ikt}.$$

This is the particular case of the Hankel functions [1]. To calculate spectrum one need to choose the normalization factor $N = (2k)^{-1/2}$. It is shown in [1] that in this case the spectrum of density fluctuations will be exactly flat, $n_s = 1$. It's clear that $\delta\phi_0$ has the form

$$\delta\phi_0 = c_+t^2 + \frac{c_-}{t}, \tag{4}$$

$c_\pm$ are constants of integration.

The general solution of the second equation from (3) is

$$\delta\psi_\kappa = C_+\left(1 - \frac{1}{\kappa t}\right)e^{\kappa t} + C_-\left(1 + \frac{1}{\kappa t}\right)e^{-\kappa t}, \tag{5}$$

with some constants $C_{\pm}$. We introduce the Darboux transformation for the (3) as

$$\begin{aligned} \delta\phi_k(t) \rightarrow \delta\phi_k^{(1)}(t;\kappa) \equiv \delta\phi_k^{(1)} &= \frac{\delta\dot{\phi}_k\delta\psi_\kappa - \delta\phi_k\delta\dot{\psi}_\kappa}{\delta\psi_\kappa}, \\ u(t) \rightarrow u^{(1)}(t;\kappa) \equiv u^{(1)} &= u + 2\frac{d^2 \log \delta\psi_\kappa}{dt^2}. \end{aligned} \tag{6}$$

It's easy to see that new function $\delta\phi_k^{(1)}$ is solution of "dressed" equation

$$\delta\ddot{\phi}_k^{(1)} + \left(k^2 + u^{(1)}\right)\delta\phi_k^{(1)} = 0, \tag{7}$$

if $\delta\phi_k$ and $\delta\psi_\kappa$ are solutions of the (3). We call $\delta\psi_\kappa$ as **prop function**. We can choose the prop function as any solution of the equation (3), for example as $\delta\phi_{\tilde{k}}$ with $\tilde{k} \neq k$, or as $\delta\phi_0$, but it is useful to choose one as (5). In this case new (integrable [2]) potential $u^{(1)}$ has the form,

$$u^{(1)} = -2\kappa^2 \frac{A^2 e^{2\kappa t} + B^2 e^{-2\kappa t} - 4AB(\kappa t)^2 - 2AB}{\left(A(\kappa t - 1)e^{\kappa t} + B(\kappa t + 1)e^{-\kappa t}\right)^2}, \tag{8}$$

so when $t \rightarrow +\infty$ then

$$u^{(1)} \rightarrow -\frac{2}{t^2} - \frac{4}{\kappa t^3} \rightarrow u(t).$$

It mean that for the enough large t_0, $u^{(1)}(t_0) \sim u(t_0)$ and therefore, the spectrum of fluctuations at this moment is the same both $u(t_0)$ and $u^{(1)}(t_0)$.

To show this we choose the normalization factor of initial $\delta\phi_k$ as $N = 1/(ik - \kappa)\sqrt{2k}$, to obtain the good asymptotic behavior for the $\delta\phi_k^{(1)}$: $\delta\phi_k^{(1)}(t \rightarrow +\infty; \kappa) = e^{ikt}/\sqrt{2k}$. We suppose that

$$0 < kt_0 \ll 1 \ll \kappa t_0, \tag{9}$$

so we are interesting of long wavelength fluctuations with wavelength $\lambda \sim 1/k \gg 1/\kappa$, at the moment $1/\kappa \ll t_0 \ll 1/k$.

Using (6), (9) we get

$$\left|\delta\phi_k^{(1)}(t_0;\kappa)\right|^2 \sim \frac{k}{2\left(k^2 + \kappa^2\right)}\left(2 + \frac{1}{(kt_0)^4} + \frac{2}{(kt_0)^3}\right).$$

To obtain $\delta_k^{(1)}$ one need to substitute $\delta\phi_k^{(1)}(t_0;\kappa)$ and $\delta\phi_0^{(1)}(t_0;\kappa)$ into the (2). The function $\delta\phi_0^{(1)}(t_0;\kappa)$ can be obtained from the (6) by the substitution $\delta\phi_k \rightarrow \delta\phi_0$ from the (4). The only k dependence is in the $\delta\phi_k^{(1)}(t_0;\kappa)$, so we omit the calculation of $\delta\phi_0^{(1)}(t_0;\kappa)$. It's clear that in main order ($\kappa \gg k$, $kt_0 \ll 1$) the amplitude $\mid \delta_k^{(1)} \mid$ does not depend on k, so we get a flat spectrum. As in the [1], we obtain this result without any brane-string physics.

To consider small deviation from flat spectrum we write

$$\mid \delta\phi_k^{(1)} \mid \sim \frac{1}{\kappa\sqrt{2}t_0^2 k^{3/2}}\left(1 + kt_0\right).$$

[2]This is because one can obtain all solutions of the (7) starting out from the known solutions of the eq. (3). This why Darboux transformation is power tool to construct many, if not all, exact soluble potential in one dimensional quantum mechanics [5].

Using (2) we get

$$n_s = 2\left(2 + \frac{k}{\mid \delta\phi_k^{(1)}(t_0) \mid}\frac{d \mid \delta\phi_k^{(1)}(t_0) \mid}{dk}\right) = 1 - kt_0,$$

so if $kt_0 \to 0$ then $n_s \to 1$ and $n_s < 1$. We get red spectrum but for the small kt_0 (large wavelengths) the deviations from the flat spectrum are small. As suggested by observation [4] we need $\mid n_s - 1 \mid < 0.1$, so $\mid kt_0 \mid < 0.1$.

A single act of dressing (6) can be iterated n times [5]. As a result one get

$$u^{(n)} = u + 2\frac{d^2}{dt^2}\log\det\begin{pmatrix} d^{n-1}\delta\psi_{\kappa_n}/dt^{n-1} & d^{n-2}\delta\psi_{\kappa_n}/dt^{n-2} & \dots & \delta\psi_{\kappa_n} \\ d^{n-1}\delta\psi_{\kappa_{n-1}}/dt^{n-1} & d^{n-2}\delta\psi_{\kappa_{n-1}}/dt^{n-2} & \dots & \delta\psi_{\kappa_{n-1}} \\ . & & & \\ . & & & \\ d^{n-1}\delta\psi_{\kappa_1}/dt^{n-1} & d^{n-2}\delta\psi_{\kappa_1}/dt^{n-2} & \dots & \delta\psi_{\kappa_1} \end{pmatrix},$$

where $\delta\psi_{\kappa_m}$ is solution of the equation (3) with $\kappa = \kappa_m$, $m = 1, .., n$. Choosing $\delta\psi_{\kappa_m}$ at the form (5) we get at the moment $t_0 \gg 1/K$, with $K = min\{\kappa_m\}$,

$$u^{(n)}(t_0) \sim u(t_0) = -2/t_0^2,$$

therefore the one-loop potential $u^{(n)}$ lead to the flat spectrum just as $u^{(1)}$ (8).

It is interesting to obtaine the classical potential $V(\phi)$ which lead to "one-loop potential" (8). We'll do it in the next section.

3 Reconstruction of potential $V(\phi)$.

To reconstruct $V(\phi)$ [3]we start with equation of motion for the field $\phi = \phi(t)$,

$$\ddot{\phi} = -V'(\phi). \tag{10}$$

Introducing new variable $\eta(t) = \dot{\phi}$ we get

$$\ddot{\eta} + V''(t)\eta = 0, \tag{11}$$

Comparing with (1) one conclude that $\eta(t) = \delta\phi_0(t)$. Solving (11) we find $V(\phi)$ in the parametric form

$$V(t) = \rho - \frac{1}{2}\eta^2(t), \qquad \phi(t) = \phi_0 + \int dt\,\eta(t), \tag{12}$$

where ρ=const is energy density and ϕ_0=const is initial value of $\phi(t)$.

Let illustrate this simple method for the $V'' = -\mu^2/t^2$. If $\mu^2 \neq 2$ then the general solution of the (11) is

$$\eta(t) = \sqrt{t}\left(C_+ t^\beta + C_- t^{-\beta}\right), \tag{13}$$

where

$$\beta = \frac{1}{2}\sqrt{1 + 4\mu^2} = \frac{1}{2}\left|\frac{3n-2}{n-2}\right|.$$

[3]The general fomulation of the reconstruction problem take place in our work [6]

The quantity $\beta = 3/2$ (flat spectrum) when $n \to \infty$ (it lead ones to the cyclic potential) and for the power-low potential with $n = 4/3$. Thus, for the $\mu^2(n) \neq 2$ one have

$$V(t) = \rho - \frac{1}{2}\left(C_+ t^2 + \frac{C_-}{t}\right)^2, \qquad \phi(t) = \phi_0 + 2t^{2/3}\left(\frac{C_+}{3+2\beta}t^{\beta} + \frac{C_-}{3-2\beta}t^{-\beta}\right), \tag{14}$$

and for the $\mu^2 = 2$

$$V(t) = \rho - \frac{1}{2}\left(C_+ t^2 + \frac{C_-}{t}\right)^2, \qquad \phi(t) = \phi_0 + \frac{C_+}{3}t^3 + C_- \log t. \tag{15}$$

In a case of general position one can't find $V(\phi)$ in non-parametric form but we can do it if one of constants $C_\pm$ is zero. Let $C_- = 0$, then

$$V_+(\phi) = \rho - g^2 \left(\phi - \phi_0\right)^{2(2\beta+1)/(2\beta+3)}, \qquad g^2 = \left(\frac{C_+^4 \left(2\beta+3\right)^{2(2\beta+3)}}{2^{6\beta+5}}\right)^{1/(2\beta+3)}.$$

We choose $\rho = 0$, $\phi_0 = 0$. Then

$$\begin{aligned} V_+(\phi) &= -g^2 \phi^{N_1(n)}, \qquad n > 2, \\ V_+(\phi) &= -g^2 \phi^{n}, \qquad n \in [1, 4/3], \\ V_+(\phi) &= -g^2 \phi^{N_2(|n|)}, \qquad n \in (-\infty, 0], \end{aligned} \tag{16}$$

where

$$N_1(n) = \frac{4(n-1)}{3n-4}, \qquad N_2(\mid n \mid) = \frac{4(\mid n \mid +1)}{3 \mid n \mid +4}.$$

It is valid $4/3 < N_1(n) < 2$ and $1 \leq N_2(n) < 4/3$. In this case (which is valid both for the (14) and (15)) the absolute value of $\phi(t)$ grows as $t \to \infty$. To obtain decreasing ones we choose $C_+ = 0$. At last we get for the (14)

$$\begin{aligned} V_-(\phi) &= -g^2 \phi^{n}, \qquad n > 0 \\ V_-(\phi) &= -g^2 \phi^{N_1(n)}, \qquad n \in [1, 4/3], \\ V_-(\phi) &= -g^2 \phi^{N_2(|n|)}, \qquad n \in (-\infty, 0], \end{aligned} \tag{17}$$

and for the (15)

$$V_-(\phi) = \rho - \frac{C_-^2}{2} e^{-2(\phi-\phi_0)/2}. \tag{18}$$

The last example is nothing but "cyclic" potential.

All known potentials (16), (17), (18) are particular cases of (14), (15) which can be reconstructed in non-parametric representation. In a case of general position, these potentials can be obtained in parametric form only. Let consider (14). If $C_+ < 0$ and $C_- < 0$ then function $\phi(t)$ is monotone decreasing function and $V(\phi)$ is single-valued function on ϕ with maximum in

$$\phi_* = \phi_0 - \frac{\mid C_- \mid}{3}\left(\frac{1}{2} + \log \frac{\mid C_- \mid}{2 \mid C_+ \mid}\right), \qquad V(\phi_*) = \rho - \frac{9}{2}\left(\frac{\mid C_+ \mid C_-^2}{4}\right)^{2/3}.$$

This decreasing potential has asymptotic behavior

$$V(\phi) \to -\frac{C_-^2}{2} e^{2\phi/|C_-|}, \qquad as \qquad \phi \to +\infty \qquad (t \to 0),$$

and

$$V(\phi) \to -\frac{C_+^{2/3}}{2} (3\phi)^{4/3}, \qquad as \qquad \phi \to -\infty \qquad (t \to +\infty).$$

If, $C_+ C_- < 0$ then the function $\phi(t)$ has one extreme point and $V(\phi)$ is two-digit function on ϕ.

Using this method one can reconstruct the classical potential $V(\phi)$ starting out with the "one-loop potential" $u^{(1)}$ (8). Dressing $\delta\phi_0$ (4) and choosing $C_+ = C_- = 1$, $c_- = 0$ we get

$$u^{(1)}(y) = \frac{d^2 V^{(1)}(\phi)}{d\phi^2} = -\frac{2\kappa \left(\sinh^2 y - y^2\right)}{\left(y \cosh y - \sinh y\right)^2},$$
$$\eta(y) = \frac{y\left(-3y \cosh y + (3 + y^2) \sinh y\right)}{y \cosh y - \sinh y}, \qquad y = \kappa t, \tag{19}$$

so using (12) one get the potential $V(\phi)$ in parametric form. It easy to see that

$$V(\phi) \sim -\frac{1}{2} (3\kappa\phi)^{4/3}, \qquad as \qquad t \to +\infty,$$

and

$$V(\phi) \sim \rho - \frac{4}{\sqrt{5}} \left(\kappa(\phi - \phi_0)\right)^{3/2}, \qquad as \qquad t \to 0,$$

therefore ρ is initial value of $V(\phi)$. Then $dV/d\phi = 2\kappa\dot{\eta} \neq 0$, so $V(\phi)$ is monotone function on ϕ. And last but not least, $\delta\phi_0$ is monotone function on time therefore $V(\phi)$ is single-valued function on ϕ.

Choosing another $c_\pm$ one can find a family of classical potentials with the same quantum potentials V'' and which lead to the same spectrum.

It is interesting to consider the last potential (which can be obtained from the (19)) from the inflation standpoint. To obtain inflation one suppose that $\dot{\phi}^2/2 \ll \mid V(\phi) \mid$ so $\rho \gg \eta^2/2$ and during inflation $V(\phi) \sim \rho$ (we suppose $\rho > 0$). Therefore $H^2 = 8\pi\rho/3M_p^2$ =const. But $k^2 > H^2$ [1]. This is because we neglect the term $3H\delta\dot{\phi}_k$ in the equation (1). On the other hand, as suggested by observation [4] we need $\mid n_s - 1 \mid < 0.1$, so $\mid kt_0 \mid < 0.1$. Introducing dimensionless T and R as $t_0 = 10^T/M_p$ and $\rho = M_p^4/10^R$ one get the inequality

$$R - 2T > 2 + \lg(8\pi/3) \sim 2.92.$$

This inequality represent restriction to possible values of t_0 and initial energy density ρ. One can see that if $t_0 \sim 10^{-35}$s then $\rho < 10^{-19} M_p^4$ so we have not normal initial conditions for the chaotic inflation.

4 Density perturbations.

As we have seen the Darboux transformation allows one to obtain a set of potentials which lead to the flat (or almost flat) spectrum. But we should distinguish between the spectrum of the scalar field and the spectrum of density perturbations [4]. An investigation of this question by Lyth [7], Brandenberger [8] and others suggests that even though the scalar field perturbations with flat spectrum can be generated by this effect, it does not lead to perturbations of metric with flat spectrum. In particularly, Brandenberger in [8] has considered the simplest potential which lead to the scale factor $a(t) \sim t^p$. For extremely slow contraction with $p \sim 0$ he get a blue spectrum with index $n \sim 3$. It is interesting to consider more general class of potentials and we'll do it in this section.

In the case of general position one can't solve the Einstein equations exactly to find the scale factor $a(t)$ but using the Darboux transformations we can construct a rich set of exact soluble potentials so this method it is interesting itself.

Let consider the Einstein equations (in the units with $c = 8\pi G = 1$)

$$\ddot{\phi} + 3H\dot{\phi} + V' = 0, \qquad H^2 = \frac{1}{3}\left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right), \tag{20}$$

so $\dot{\phi}^2 = -2\dot{H}$. Introducing $\psi(t) = a(t)^3$ one get $\ddot{\psi} = 3V\psi$. If we add positive cosmological term $\Lambda > 0$ then this equation can be written as the Schrödinger equation

$$\ddot{\psi} = \left(u + \lambda^2\right)\psi, \tag{21}$$

where $u(t) = 3V(\phi(t))$, $\lambda^2 = 3\Lambda$. We denote the scale factor and the Hubble "constant" in the universe with $\lambda = 0$ as $a = a(t) = \psi^{1/3}$, $H = H(t)$ whereas the same in the universe with $\lambda \neq 0$ will be $\tilde{a} = \tilde{\psi}^{1/3}$ and $\tilde{H}$.

We suppose that in our universe the $\lambda^2 = 0$. Some researchers are suggested that the reason of the recently discovered accelerated expansion of the universe is the nonzero cosmological term. Indeed, it is easy to formulate usual inflationary theory to obtain this accelerated expansion. To do it one can add a small constant term $V_0 > 0$ to the potential $V(\phi)$ [9] [5]. If cosmological constant $V_0 \sim 10^{-120}$ (in Planck units) then one get the present accelaration [10], but is not terrifically because it is not clear why should V_0 be so small? This approach return us to the old mystery of vacuum energy [11] and this is why we suppose that $\lambda^2 = 0$ in our universe. The solutions of (20) with nonzero cosmological term will be necessary for the Darboux transformation (see below).

The Darboux transformation for the (21) has the form

$$\psi \to \psi^{(1)} = (\sigma - \tilde{\sigma})\,\psi, \qquad u \to u^{(1)} = u - 2\dot{\tilde{\sigma}},$$

where $\sigma = d\log\psi/dt$, $\tilde{\sigma} = d\log\tilde{\psi}/dt$. One get

$$H^{(1)} = -\tilde{H} + \frac{\lambda^2}{9(\tilde{H} - H)}, \tag{22}$$

therefore the dressed scale factor $a^{(1)}(t)$ has the form

$$a^{(1)} = \frac{const}{\tilde{a}} \exp\left[-\frac{\lambda^2}{9}\int dt \left(\frac{d}{dt}\log\frac{\tilde{a}}{a}\right)^{-1}\right]. \tag{23}$$

At last, dressed potential $V^{(1)}(\phi^{(1)})$ can be written in parametrical form,

$$\begin{aligned} \phi^{(1)}(t) &= \phi_0^{(1)} - \sqrt{\frac{2}{3}}\int dt \sqrt{\frac{\lambda^4}{9(H - \tilde{H})^2} + \lambda^2\frac{H + \tilde{H}}{H - \tilde{H}} + 3\dot{\tilde{H}}}, \\ V^{(1)}(t) &= V(t) - 2\dot{\tilde{H}}, \end{aligned} \tag{24}$$

[4] I'd like to thank prof. A. Linde who drew my attention to this circumstance.

[5] Cosmology with negative potentials were considered in [10].

where $\phi_0^{(1)}$ is constant.

This method of construction of exact soluble potentials is valid both for usual inflation cosmology and for the effective field theory of the Ekpyrotic/Cyclic Universe. In the last case the scalar field ϕ represents the position of the bulk brane in the fifth dimensions. Following Brandenberger one consider the scale factor $a(t) \sim (-t)^p$ (time is negative). If $\lambda^2 = 0$ then

$$u(t) = \frac{3p(3p-1)}{t^2}.$$

The solution of (21) with $\lambda^2 \neq 0$ has the form

$$\tilde{\psi} = \sqrt{-t}\left(c_1 K_\nu(-\lambda t) + c_2 I_\nu(-\lambda t)\right), \tag{25}$$

where $\nu = \pm(1-6p)/2$, $c_1 > 0$ and $c_2 > 0$ are constants, K_ν and I_ν are modified Bessel function (if $\Lambda < 0$ then the solution of the (21) will be expressed via Hankel functions but the final result will be the same), $t < 0$ and we choose $\nu > -1$ to obtain real and positive solution (25). For the small λt one get

$$\tilde{\psi} = \frac{c_1}{\sqrt{2\lambda}}\Gamma(\nu)\left(\frac{-\lambda t}{2}\right)^{-\nu-1/2} + c_2\sqrt{\frac{2}{\lambda}}\Gamma(\nu+1)\left(\frac{-\lambda t}{2}\right)^{-\nu+1/2}. \tag{26}$$

Using (23), (26) and well known relations between modified Bessel functions and it's derivatives [12] we have $a^{(1)} \sim (-t)^{p^{(1)}}$. The power $p^{(1)}$ is depend on initial power p. For example, if $1/3 < p < 1/2$, then $p^{(1)} = p - 1/3$, so $0 < p^{(1)} < 1/6$, and we have blue spectrum with index

$$n = 1 + \frac{2}{1 - p^{(1)}} = 1 + \frac{6}{4 - 3p}.$$

If $p \to 1/3$ then we get the extremely slow contraction and the spectrum with index $n \sim 3$.

Another values of $p^{(1)}$ can be obtained starting out from the another choices of p. The general conclusion is that all new potentials leads to the old result of Brandenberger. In other words we have not the flat spectrum of perturbations of metric.

5 Conclusion.

Up to know, Darboux transformations didn't used in cosmology (by contrast with quantum mechanics and the theory of integrable PDE). In this paper we are used only simple example of Darboux transformation. One can construct infinite set of potentials which lead to the flat spectrum with arbitrary multiple precision. We can do it without any brane-string physics and we don't need consider field theory with power-low potential when $n > 40$.

On the other hand, it is valid only for the spectrum of the scalar field and, to all appearances, don't valid for the spectrum of density perturbations. Using Darboux transformations, one can construct infinite set exact soluble cosmologies with flat spectrum of the scalar field perturbations and, at the same time, with "essentially blue" spectrum of perturbations of metric.

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